Stochastic modelling of soil erosion and deposition

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Abstract
The rate of erosion is the product of the concentration of active soil aggregates and their mean vertical velocity. The sediment concentration for a given time is proportional to the probability of detachment of first-time mobilized soil aggregates and to the probability of the deposited sediments' redetachment. The probability of soil aggregate activation is equal to the probability of the driving forces exceeding the resistance forces. These forces are stochastic variables, and any excess value of the driving forces above the resistance forces is a probability function of these stochastic variables. Five characteristics are used as stochastic variables: flow velocity; soil cohesion; aggregate size for the native soil and deposited sediments; and soil consolidation (or soil fatigue). The vertical velocity of soil aggregates at the moment of detachment is derived from the momentum continuity equation for the stable particle at the flow bed. The proposed theory explicitly describes differences in types of relationships between detachment rates and flow velocities; the different shapes of the probability density curves of soil properties (cohesion, aggregate size and soil consolidation) cause this difference. The detachment rate increased with flow velocity more rapidly for more consolidated soils with high cohesion, and large aggregates.

Introduction
Soil erosion rate is controlled by water flow parameters (velocity, depth and turbulence) and soil texture (mechanical pattern and protection by vegetation). These characteristics are combined in an equation of mass conservation (1), which can be written in the following simplified form:

$$\frac{\partial QC}{\partial X} = M_0 W - k_0 C V_f W \qquad (1)$$

Here Q = water discharge (m^3/s); C = mean volumetric sediment concentration; X = longitudinal coordinate (m); M_0 = upward sediment flux (m/s); W = flow width (m); V_f = soil aggregates fall velocity in the turbulent flow (m/s); k_0= the coefficient to transform mean sediment concentration into near-bed sediment concentration. The left part of equation (1) defines the sediment budget in the flow. The right part of (1) defines the sediment flux: the first term is upward flux, and the second is downward flux.
The upward sediment flux M_0 (or detachment rate) is the product of the concentration (C_Δ) of active soil aggregates in the bed layer with the thickness Δ and the mean vertical velocity of soil aggregates ($U_\uparrow$):

$$M_0 = C_\Delta U_\uparrow \qquad (2)$$

Both the concentration of active soil aggregates and the vertical velocity of these aggregates are stochastic variables in the probabilistic field of driving turbulent flow and resisting soil texture and vegetation cover. The main goal of a stochastic approach to soil erosion is to estimate the main parameters of this probabilistic field and to find the theoretical relationship with the soil erosion rate. H. A. Einstein (1942) first used this approach for the transport of non-cohesive sediments. Mirtskhoulava (1988), Nearing (1991), Wilson (1993a, 1993b) and Lisle et al. (1998) formulated probabilistic concepts of detachment for cohesive soils. The following theoretical stochastic description of soil erosion is an expansion of the models listed above, with the significant inclusion of stochastic variables that govern this complicated process.

Bed concentration of active soil aggregates
Concentration is the ratio between the volume of active aggregates V_a and the volume V of the bed layer (with the thickness Δ and the unit area S): $C_\Delta = V_a/(\Delta * S)$. The volume of active aggregates can be written as the product of the number of active aggregates N and their mean volume V_m: $V_a = NV_m$. The unit area of the bed layer can be presented as the product of the number of aggregates M, exposed to the flow bed on the unit area, and their mean area S_m: $S = MS_m$. Therefore, active aggregates concentration can be represented as:

$$C_\Delta = \frac{NV_m}{MS_m \Delta} \qquad (3)$$

The ratio N/M is the probability (P_d) of soil aggregate detachment for a given unit time $dt = \Delta / U_\uparrow$, and the ratio V_m/S_m is some measure of the mean soil aggregate height D_m. Therefore

$$C_\Delta = \frac{D_m}{\Delta} P_d \qquad (4)$$

When the active bed layer has thickness Δ equal to aggregate height D_m, the active aggregates concentration in the bed layer is equal to the probability of detachment. That is the most simple case of a rather low erosion rate of cohesive soil, when only one particle-thick layer can be eroded during the unit time *dt*.
Following Hairsine and Rose (1991), the soil aggregates can be mobilized from the native soil bed and redetached from deposited layer. In this case the volume of active aggregates is equal to the sum of the volume of first time mobilized active soil aggregates N_1V_{m1} and the volume of redetached aggregates N_2V_{m2}. Again, the unit area of the bed layer is the sum of the area of exposed native soil M_1S_{m1} and the area of deposited aggregates M_2S_{m2}.

$$C_\Delta = \frac{N_1V_{m1} + N_2V_{m2}}{M_1S_{m1}\Delta_1 + M_2S_{m2}\Delta_2} \qquad (5)$$

This formula can be transformed to

$$C_\Delta = \frac{N_1V_{m1}}{M_1S_{m1}\Delta_1\left(1+\dfrac{M_2S_{m2}\Delta_2}{M_1S_{m1}\Delta_1}\right)} + \frac{N_2V_{m2}}{M_2S_{m2}\Delta_2\left(1+\dfrac{M_1S_{m1}\Delta_1}{M_2S_{m2}\Delta_2}\right)}$$

and to

$$C_\Delta = k_1C_{\Delta 1} + k_2C_{\Delta 2} = k_1\frac{D_{m1}}{\Delta_1}P_{d1} + k_2\frac{D_{m2}}{\Delta_2}P_{d2} \qquad (6)$$

Here indices 1 and 2 relate to aggregates in the native soil and to deposited soil aggregates, respectively. The coefficients k_1 and k_2 represent the ratios between the volumes of the exposed aggregates in native soil and deposited layer, and the whole bed layer volume. Therefore $k_1=1-k_2$. The ratio k_2 can be established from the continuity equation (1). The volume of the deposited layer is the difference between the volume of deposition and the volume of erosion of the deposited sediments:

$$M_2S_2\Delta_2 = \left(MS_mk_0CV_f - M_2S_2C_{\Delta 2}U_{\uparrow 2}\right)dt\,. \qquad (7)$$

As $dt=\Delta/U_\uparrow$ and $C = \dfrac{C_\Delta U_\uparrow W - \dfrac{\partial Q_s}{\partial X}}{k_0V_fW} = \dfrac{k_3U_\uparrow}{k_0V_f}C_\Delta$ then

$$k_2 = \frac{M_2S_2\Delta_2}{MS_m\Delta} = k_0C\frac{V_f}{U_\uparrow} - k_2C_{\Delta 2}\frac{U_{\uparrow 2}}{U_\uparrow} = k_3C_{\Delta 1} - k_2k_3C_{\Delta 1} + k_2k_3C_{\Delta 2} - k_2C_{\Delta 2}\frac{U_{\uparrow 2}}{U_\uparrow}$$ and finally

$$k_2 = \frac{k_3C_{\Delta 1}}{1 + k_3C_{\Delta 1} - k_3C_{\Delta 2} + C_{\Delta 2}\dfrac{U_{\uparrow 2}}{U_\uparrow}}\,. \qquad (8)$$

The probability of soil aggregate detachment is equal to the probability of the excess of driving forces above resistance forces in the flow. Driving forces are drag force (F_d), lift force (F_l), negative turbulent dynamic pressure (F_{dp}), and pore water pressure (F_{pw}). Resistance forces are submerged weight (F_w), friction force (F_f), static pressure (F_{sp}), positive turbulent dynamic pressure (F_{dp}) and cohesion (F_c). After Mirtskhoulava (1988) and Borovkov (1989)

$$F_d = C_R\rho S_d\frac{U^2}{2} \qquad (9) \qquad\qquad F_l = C_y\rho S_u\frac{U^2}{2} \qquad (10)$$

$$F_{dp} = 3.5\lambda\rho S_b\frac{U_m^2}{2} \qquad (11) \qquad\qquad F_{pw} = g\rho S_bz_p \qquad (12)$$

$$F_w = gV_u\left(\rho_s - \rho\right) \qquad (13) \qquad\qquad F_f = f_t\,gV_u\left(\rho_s - \rho\right) \qquad (14)$$

$$F_{sp} = g\rho S_bd \qquad (15) \qquad\qquad F_c = C_0S_b \qquad (16)$$

Here C_R is the coefficient of drag resistance; C_y is the coefficient of uplift; U is the actual near-bed flow velocity, and U_m is its mean value; λ is the coefficient of hydraulic resistance; S_d is the cross-section area of soil aggregate, perpendicular to flow; ρ_s and ρ are the soil aggregate density (containing pores) and water density respectively; S_u is the cross-section area of the soil aggregate, parallel to the flow (vertical projection); S_b is the area of the soil aggregate that is solid with native soil and other aggregates; z_p is capillary pressure height; f_t is the friction coefficient; d is water depth; C_0 is soil cohesion. Equations (9–10) and (13–14) are related to all the soil aggregates at the flow bed, equations (11–12) and (15–16) can be used only for those soil aggregates, which are solid (by cohesion) with the native soil and/or with the other aggregates.

A probability of detachment is greater than zero, when the module of sum of the driving forces is more than the sum of resistance forces:

$$\left| F_{d1} + F_{l1} + F_{p1} \mp F_{dp1} - F_{w1} - F_{f1} - F_{sp1} - F_{c1} \right| > 0 \qquad (17_1)$$

$$\left| F_{d2} + F_{l2} + F_{w2} - F_{f2} \right| > 0 \qquad (17_2)$$

Here indices 1 and 2 relate forces to the aggregates in the native soil and to loose deposited aggregates, respectively. After dividing (17) over $\frac{1}{2}\rho S_u\left(kC_R + C_y\right)$, where $k = S_d/S_u$, it is transformed into

$$\Psi_1 = U^2 + k_{pw} z_p \frac{S_b}{S_{u1}} \mp k_{dp}\lambda \frac{S_b}{S_{u1}} U_m^2 - k_{wf} D_{m1} \frac{(\rho_s - \rho)}{\rho} - k_{sp} d \frac{S_b}{S_{u1}} - k_c \frac{C_0}{\rho} \frac{S_b}{S_{u1}} > 0 \qquad (18_1)$$

$$\Psi_2 = U^2 - k_{wf} D_{m2} \frac{(\rho_s - \rho)}{\rho} > 0 \qquad (18_2)$$

The values of the coefficients can be obtained from Mirtskhoulava (1988) and Borovkov (1989):

$$k_{pw} = \frac{2g}{\left(kC_R + C_y\right)} \approx 40; k_{dp} = \frac{3.5}{\left(kC_R + C_y\right)} \approx 7; k_{wf} = \frac{2(1 + f_t)g}{\left(kC_R + C_y\right)} \approx 42;$$
$$k_{sp} = \frac{2g}{\left(kC_R + C_y\right)} \approx 40; k_m = \frac{24}{kC_R + C_y} \approx 48; k_c = \frac{2}{\left(kC_R + C_y\right)} \approx 4. \qquad (19)$$

Driving and resistance forces are stochastic variables, and their sum Ψ has some stochastic distribution with the probability density function p_Ψ. Therefore, the probability of detachment P_d can be calculated with the formula:

$$P_{d1} = \int_0^\infty p_{\Psi 1} d\Psi_1 \qquad (20_1)$$

$$P_{d2} = \int_0^\infty p_{\Psi 2} d\Psi_2 \qquad (20_2)$$

The vertical velocity of soil aggregates

The vertical velocity of soil aggregates is the second component of the formula (2) for the detachment rate calculation. The moment of aggregate detachment acceleration can be derived from the expression

$$D_m \frac{(\rho_s - \rho)}{2\rho} \frac{\partial U_\uparrow^2}{\partial z} = \Psi \qquad (21)$$

In the bed layer with thickness Δ, an aggregate accelerates from zero velocity to its maximum value, $U_\uparrow$. The integral of (21) gives a simple expression for the near bed vertical velocity of aggregates:

$$U_\uparrow = \sqrt{\Psi \frac{\Delta}{D_m} \frac{2\rho}{(\rho_s - \rho)}} \qquad (22)$$

In turbulent flow with random vertical velocity, its mean value in the field of positive forces $\Psi>0$ may be calculated with the formula:

$$\overline{U_\uparrow} = \frac{\int_0^\infty p_\Psi \sqrt{\Psi \frac{\Delta}{D_m} \frac{\rho}{(\rho_s - \rho)}} d\Psi}{\int_0^\infty p_\Psi d\Psi} \qquad (23)$$

Discussion

Theoretical analysis of the stochastic mechanics of soil aggregate erosion in water flow shows that in the field of random driving and stabilizing forces, the detachment rate can be calculated as product of Eqs ($20_{1\text{-}2}$) and (23):

$$M_0 = (1 - k_2)\frac{D_{m1}}{\Delta_1}\int_0^\infty p_{\Psi 1}\sqrt{\Psi_1 \frac{\Delta_1}{D_{m1}} \frac{\rho}{(\rho_s - \rho)}} d\Psi_1 + k_2 \frac{D_{m2}}{\Delta_2}\int_0^\infty p_{\Psi 2}\sqrt{\Psi_2 \frac{\Delta_2}{D_{m2}} \frac{\rho}{(\rho_s - \rho)}} d\Psi_2 \qquad (24)$$

where

$$k_2 = \frac{k_3 \frac{D_{m1}}{\Delta_1} \int_0^\infty p_{\Psi 1} d\Psi_1}{1 + k_3 \frac{D_{m1}}{\Delta_1} \int_0^\infty p_{\Psi 1} d\Psi_1 - \left(k_3 - \frac{U_{\uparrow 2}}{U_\uparrow} \right) \frac{D_{m2}}{\Delta_2} \int_0^\infty p_{\Psi 2} d\Psi_2} . \qquad (25)$$

The probability of a function of stochastic variables can be calculated if the probabilities of the individual variables are known (Gnedenko, 1954). A probability of product *Z* of stochastic variables *X* and *Y* is derived from the integrals:

$$p_Z(Z) = -\int_{-\infty}^{0} X^{-1} p_X(X) p_Y\left(\frac{Z}{X}\right) dX + \int_{0}^{\infty} X^{-1} p_X(X) p_Y\left(\frac{Z}{X}\right) dX \qquad (26)$$

A probability of sum *Z* of stochastic variables *X* and *Y* is derived from the function:

$$p_Z(Z) = \int_{-\infty}^{\infty} p_X(X) p_Y(Y) dX = \int_{-\infty}^{\infty} p_X(X) p_Y(Z - X) dX . \qquad (27)$$

We shall work out a simplified case, where five characteristics are taken as stochastic variables: velocity *U*, cohesion C_h, aggregate size D_{m1} (native soil), D_{m2} (deposited aggregates), and soil consolidation $I_s = S_b/S_u$, and all others are parameters. Therefore the probability density functions for stochastic variables must be estimated theoretically or experimentally.

A probability density function p_U for actual near bed velocity *U* with mean value U_m and standard deviation σ_U is often described by the normal distribution (Mirtskhoulava, 1988). Then the frequency of $z = U^2/\sigma^2$ will be defined by first order non-central χ^2 distribution (Pugachev, 1979). Borovkov (1989) showed that σ_U is related to dynamic velocity: $\sigma_U = 3.0\ u_*$.

The distribution density of soil aggregate size (both within a sample of native soil and for deposited aggregates) generally fits a lognormal distribution with the parameters related to mean aggregate diameter D_m and its standard deviation σ_D.

The soil consolidation and cohesion are attributed only for the aggregates of native soil. The ratio of the soil aggregate area S_b, where aggregate is solid with the native soil or other aggregates, to the aggregate vertical projection area S_u: $I_s = S_b/S_u$ is the measure of the soil consolidation. The difference between these two areas is the area of micro cracks that cut loose individual aggregate from native soil. Such micro-cracks filled with ice are often formed in the frozen soil. The relative volume of micro cracks is approximately the difference between bulk soil porosity and structural within-aggregate porosity. Soil consolidation is opposite to soil fatigue, generated in the soil under a dynamic action of turbulent flow (Mirtskhoulava, 1988), and mainly due to flow velocity oscillation and dynamic pressure rapid change. Its distribution depends on soil texture, cohesion and the intensity of turbulent oscillations. An overview of laboratory and field experiments of Mirtskhoulava (1988) shows that for a wide range of different soils, $(I_s)_{mean}$ has an asymmetrical distribution. Beta-distribution will therefore be used in further calculations. It is evident that soil consolidation or fatigue, as defined above, needs further investigation.

Analysis of the laboratory data of Mirtskhoulava (1988) shows that a gamma-distribution can be used to describe the distribution of cohesion within a sample of soil. Mirtskhoulava's data also showed that the coefficient of variation $C_v = \sigma_C/C_m$ for this distribution is constant and equals ~ 0.2 for wide range of soil characteristics. In this case, the distribution curve for actual cohesion is determined only by one parameter: mean cohesion of soils.

The analytical form of (24–25) is rather complicated, and has to be solved numerically with a given set of input data. A FORTRAN program (available from the author) was written for these calculations. The input data consisted of mean bed velocity U_m, mean soil cohesion C_0, mean soil consolidation I_s, mean aggregate diameter $D_{m1\text{-}2}$ and its standard deviation $\sigma_{D1\text{-}2}$. The hydraulic resistance coefficient λ, flow depth *d*, pore water pressure height *z*, and aggregate density (with porosity) ρ_s, must also be known. Numerical experiments were carried out to analyse the influence of these five stochastic factors on the detachment rate. The range of flow bed velocity was 0.1–2.0 m/s, the range of cohesion was 1–60 kPa, soil consolidation ranged from 0.1 to 0.9, aggregate mean size in the natural soil varied from 1 to 10 mm. The deposited aggregate mean size was calculated with the help of the continuity equation. Other parameters were constant: flow depth was 0.01 mm; pore pressure height was 0.001 m; the hydraulic resistance coefficient was 0.01; aggregate density was 1600 kg/m^3; and aggregate size standard deviation in the native soil was 0.3D_m.

The detachment rate increased with flow velocity. This increase in erosion rate cannot be described with an often-used simple power function $M_0 \sim U_m^n$. Theoretical calculations showed that in relatively low velocities, the detachment rate increases more rapidly than in relatively high velocities. A similar effect was described by Nearing et al. (1997) on the basis of observations of empirical soil erosion measurements. The current theory explains this phenomenon. The detachment rate increase is controlled by soil cohesion, by aggregate size and, very significantly, by soil

consolidation. The detachment rate increased more rapidly with flow velocity for more consolidated soil with high cohesion, large aggregates, and high soil consolidation. Decrease of soil consolidation and aggregate size led to a decrease of the exponent in power law of detachment rate versus flow velocity.
Calculations also show great differences in the type of soil erosion in the relatively high and relatively low flow velocities. When flow velocities are relatively high and driving forces increase significantly over stabilizing forces, soil properties (cohesion, aggregate size, soil consolidation) are less important in determining soil-erosion rate. This implies that the time and space random variability of these factors, which always exist in natural conditions, will not lead to major changes in erosion rate. When flow velocities are relatively low and driving forces only slightly increase over stabilizing forces, soil properties are very important in determining soil erosion rates. Even the small time and space random variability of these properties may lead to significant changes in erosion rate.
To verify the theoretical results, 33 sets of data, collected for WEPP model (Elliot et al., 1989) were used. The detachment rate, hydraulic flow parameters, soil cohesion and aggregate size were published for these sites. Soil consolidation was unknown for all data sets. Optimisation calculations were performed to estimate unknown soil consolidation values. The same procedure of optimization was used by Wilson (1993b) for unknown parameters of similar type. The stochastic approach helps explain the rather broad range of exponent values in the power law between detachment rate and velocity, obtained in the field experiments (Fig. 1a) with the rather good correlation between observed and calculated values of the detachment rate (Fig. 2b).

Conclusion
The stochastic method of detachment-rate estimation is based on calculation of the probability of excess of driving forces above resistance forces in the flow that erodes cohesive soil. The explicit relationships of the hydraulic characteristics of the flow (actual flow velocity, water depth, dynamic pressure) and the mechanical properties of the soil (cohesion and consolidation) with the soil aggregates detachment rate make possible an explanation of the difference in types of relationship between detachment rate and flow velocity (shear stress, stream power) for different soils. In high-flow velocities, when driving forces increase significantly above stabilizing forces, the rate of erosion increase with flow velocity is relatively low. The influence of the variability of soil properties (cohesion, aggregate size, soil consolidation) is also less important in determining the soil erosion rate of high relative flow energy. This may be the main reason for the greater predictive capability of existing soil erosion models for high-energy events. With low flow velocities and with driving forces only slightly increased above stabilizing forces, erosion rates speedily increase with flow velocity. Soil property variability causes significant changes in soil erosion rates, and this influence grows with the increase of soil cohesion, consolidation and soil aggregates size. Even minor spatial and temporal random variability of these properties may lead to significant changes in erosion rate. This may be the reason why rather high errors in soil erosion calculations are found even with detailed physical based models for low erosion rates.

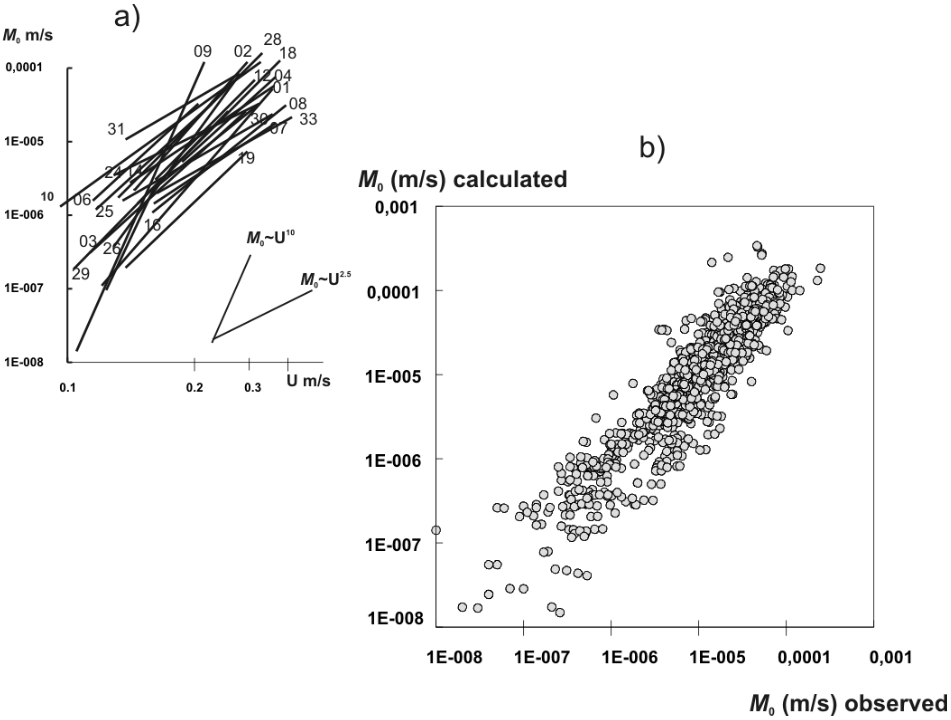

Figure1. The relationships (a) between the detachment rate M_0 and mean flow velocity U, obtained by Elliot et al. (1989) in field experiments with 33 soils over USA, showing the broad range of exponents in the power law, and the comparison (b) of observed detachment rate values with those calculated with the formulas (24–25)

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