Probability distribution function approach in stochastic modelling of soil erosion

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Abstract Stochastic modelling of soil erosion is based on calculation of the probability of soil particle detachment, which is the probability of excess of driving forces above resistance forces. These probability calculations require the probability distribution functions (PDFs) for the main hydrodynamic and soil structure characteristics, estimated experimentally or theoretically. The field of hydrodynamic forces (flow velocities and pressure distribution though space and time) is calculated with Large Eddy Simulation. Soil structure is estimated in terms of Kolmogorov's probabilistic approach to soil failure and aggregation. The PDF approach explicitly describes the process of soil erosion and gives a theoretical explanation of the great diversity in empirical relationships between erosion rate and main erosion factors.

Key words large eddy simulation; probability of detachment; probabilistic soil failure; soil erosion; stochastic modelling

INTRODUCTION

In spite of its major significance for strategic estimates and predictions related to many aspects of human activity, water erosion theory for cohesive soils is still largely undeveloped. For many years, efforts have mainly focused on the development of the empirical predictive relationships, based on data collected in areas with different climatic and land-use conditions (Merritt *et al.*, 2003). The most successful example is the so-called Universal Soil Loss Equation (Wischmeier & Smith, 1965). More recently, erosion models that address causative aspects have appeared, providing strong competition for the purely empirical models. An important step in this development was a paper by Foster & Meyer (1972), in which the sediment-budget approach to erosion modelling was suggested and developed. However, these models are still semi-theoretical or semi-empirical, as simplified stream power (or bed shear stress) relationships are used to describe such complicated phenomenon as the rate of erosion, while the whole complexity of soil resistance to erosion is expressed by simplistic erodibility coefficients.

Purely empirical and semi-empirical models do not promise much progress in soil erosion predictions and simulations. A new generation of theoretical erosion models is urgently needed that can account for the stochastic nature of soil erosion, based on mechanistic representations of the key physical processes. Recent achievements in deterministic-stochastic hydrodynamics of shallow rough-bed flows (Nikora *et al.*, 2001) make the development of such an approach feasible. Here we present a stochastic concept first, and then describe potential modelling approaches, which should provide necessary parameterization for bulk stochastic models and also give a deeper insight into erosion processes.

STOCHASTIC CONCEPT IN SOIL EROSION MODELLING

The rate of soil erosion can be estimated in two main ways (Sidorchuk, 2004): by multiplication of sediment concentration by soil particle vertical velocity (velocity–concentration); and by spatiotemporal averaging of unstable sediment particle volume on the time period for detachment (double–averaging). Within the first approach, two stochastic variables are required to calculate the rate of soil aggregates detachment *DER*: unstable aggregate concentration in the bed surface layer C_Δ and mean vertical velocity of unstable aggregates $U_\uparrow$:

$$DER = C_\Delta U_\uparrow \quad (1)$$

Bed concentration of unstable soil aggregates of a given size

The bed concentration of unstable aggregates is the ratio of the volume V_u of unstable aggregates and the whole volume V of the bed surface layer: $C_\Delta = V_u/V$. The volume of unstable aggregates can be written as the product of the number of unstable aggregates N and the mean unstable aggregate volume V_a:$V_u = NV_a$. The volume of aggregates in a surface layer can be presented as the product of the number of aggregates M, exposed to the flow on the unit area, and their mean volume V_{sm}:$V = MV_{sm}$. Therefore the concentration of unstable aggregates is:

$$C_\Delta = NV_a/MV_{sm} \quad (2)$$

The ratio N/M is the probability (P_{DER}) of soil aggregate detachment, and the ratio V_a/V_{sm} is a measure k_D of those soil aggregates' relative size. Therefore:

$$C_\Delta = k_D P_{DER} \quad (3)$$

An equation of this type was proposed by H. Einstein (1937), and is of main significance in the stochastic approach to erosion calculation. As sediment concentration appears to be proportional to the probability of detachment, the main goal of a stochastic methodology in soil erosion is to estimate this probability. The main method is to find the parameters of the probabilistic field of driving and resistance forces. Then, the probability of soil aggregate detachment can be found with the use of the condition of soil aggregate instability on the flow bed.

Soil aggregate instability

Soil aggregate detachment occurs because driving hydrodynamic forces exceed gravitational, hydrodynamic and geo-mechanical resistance and stabilizing forces. The main driving forces are form drag force (F_{fd}), wave drag force (F_{wd}), lift force (F_l), negative turbulent dynamic pressure (F_{dp}), pore water pressure (F_{pw}), and tangent component of submerged weight (F_{wt}). Resistance and stabilizing forces are normal components of submerged weight (F_{wn}), static pressure (F_{sp}), and positive turbulent dynamic pressure (F_{dp}). Mirtskhoulava (1988) and Lawrence (2000) showed that:

$$F_{fd} = C_R \rho S_d \frac{U^2}{2} \quad (4)$$

$$F_{wd} = \rho S_d C_{Rw} \frac{k_e D U^2}{2d} \tag{5}$$

$$F_l = C_y \rho S_a \frac{U^2}{2} \tag{6}$$

$$F_{dp} = 3.5\lambda\rho S_b \frac{U_m^2}{2} \tag{7}$$

$$F_{pw} = g\rho S_p z_p \tag{8}$$

$$F_{wt} = V_a(\rho_s - \rho)g\sin\beta \tag{9}$$

$$F_{wn} = V_a(\rho_s - \rho)g\cos\beta \tag{10}$$

$$F_{sp} = g\rho S_b d \tag{11}$$

where C_R is the coefficient of drag resistance; C_y is the coefficient of uplift; C_s is the coefficient of static drag; C_{Rw} is the coefficient of wave drag; U is the actual near-bed flow velocity, and U_m is its mean (time averaged) value; λ is the coefficient of hydraulic resistance; S_d is the cross-sectional area of the soil aggregate, perpendicular to the flow; V_a is the volume of the soil aggregate; S_a is the cross-sectional area of the soil aggregate, parallel to the flow (vertical projection); S_b is the area of the soil aggregate that is attached to other aggregates; S_p is the area of pores; D is the aggregate diameter; z_p is capillary pressure height; β is the angle of flow bed local inclination; k_e is the exposure of a soil aggregate and d is water depth.

Finally, there is a complex system of geo-mechanical and electro-chemical forces, defined as soil cohesion (F_c). This is a reactive force; its magnitude and direction are determined by the sum of all the above-listed active forces. Its maximum magnitude is:

$$F_c = C_0 S_b \tag{12}$$

where C_0 is soil cohesion.

Detachment occurs when the sum of the driving forces is larger than the sum of the resistance forces. For simplification, only normal components of the forces are analysed further. We define $\Theta_\uparrow$ as the inertial force that results from the force balance, normalized by $1/2\rho S_a C_y$.

$$\Theta_\uparrow = U^2 + k_{pw} z_p \frac{S_b}{S_a} \mp k_{dp}\lambda \frac{S_b}{S_a} U_m^2 - k_w D \frac{(\rho_s - \rho)}{\rho} - k_{sp} d \frac{S_b}{S_a} - k_c \frac{C_0}{\rho}\frac{S_b}{S_a} > 0 \tag{13}$$

Driving and resistance forces are stochastic variables and, consequently, the function $\Theta_\uparrow$—the condition of instability—has some stochastic distribution (within a spatial/temporal "window" at the flow bed surface) with the PDF p_Θ. The PDF of the function of stochastic variables can usually be calculated when the PDFs of those stochastic variables are defined.

The probability of the detachment of the aggregate P_{DER} is the sum of p_Θ for all positive values of $\Theta_\uparrow$:

$$P_{DER} = \int_0^\infty p_\Theta d\Theta \tag{14}$$

The vertical velocity of soil aggregates

The vertical velocity of soil aggregates is the second component of the formula (1) for the detachment rate calculation. The acceleration along the vertical co-ordinate z at the moment of an aggregate detachment can be derived from the second Newton law, written for the normal component of forces (see equation 13) and aggregate acceleration:

$$V_a \frac{\rho_s}{2} \frac{\partial U_\uparrow^2}{\partial z} = \frac{1}{2} \rho S_a C_y \Theta_\uparrow \tag{15}$$

In a bed layer with thickness D, an aggregate accelerates from zero velocity to its maximum value, $U_{\uparrow \max}$. The soil integrity $I_s = S_b/S_a$ decreases from maximum I_{s0} to zero within the bed surface layer (at the distance equal to D).

$$I_s = I_{s0} - \frac{I_{s0}}{D} z \tag{16}$$

The integral of (15) with (13) and (16) gives a parabolic expression for actual vertical velocity of an aggregate in a bed layer:

$$\begin{aligned} U_\uparrow^2(z) = {} & \frac{\rho C_y}{\rho_s D} \left(U^2 + k_{pw} z_p I_{s0} \mp k_{dp} \lambda I_{s0} U_m^2 - k_c \frac{C_0}{\rho} I_{s0} - k_{sp} d I_{s0} - k_w \frac{(\rho_s - \rho)}{\rho} D \right) z \\ & + \frac{\rho C_y}{2 \rho_s D} \left(k_{pw} z_p \mp k_{dp} \lambda U_m^2 + k_{sp} d + k_c \frac{C_0}{\rho} \right) \frac{I_{s0}}{D} z^2 \end{aligned} \tag{17}$$

Averaged in the bed layer, the vertical aggregate velocity $U_{\uparrow m}$ can be easily calculated from (17), not presented here because of the great length of the expression.

In the field of random forces the vertical velocity for an aggregate is a random variable with PDF $p_{U\uparrow}$. Its mean value:

$$\overline{U_\uparrow} = \int_0^\infty p_{U\uparrow} U_{\uparrow m} dU_\uparrow \tag{18}$$

is combined with P_{DER} (14) to give the expression for the aggregate detachment rate calculation (1). These calculations require probability distributions of hydrodynamic and soil characteristics, which can be estimated both experimentally and theoretically.

MECHANISM-BASED APPROACHES IN NUMERICAL MODELLING OF PDF

Hydrodynamic characteristics

To underpin and complete the probabilistic approach for modelling erosion and sedimentation processes, we require a physically based model. This mechanistic hydrodynamic model will provide a sound representation of the fluid flow in sedimentary environments. Shallow flow and changeable rough surfaces are characteristic of such environments. To be consistent with our objective of considering fundamental principles of turbulence hydrodynamics and soil physics, simple configurations of erosion should be addressed first, with the intent that the modelling can be developed to incorporate higher degrees of complexity and additional factors. In particular, we intend to model four situations, in which we consider: flow, flow and rain (as a significant source of energy to the system), flow and erosion, flow and rain and erosion, respectively. The magnitudes of the forces (4) to (12) and, therefore, probability of detachment, are different for these four situations. All scenarios involve modelling flow past complex boundary conditions, which is therefore an important criterion for potential models. The fluid model must therefore be able to accommodate a high level of spatial complexity. Further, there is a dynamic fluid–solid interaction that takes place in these environments: the pressure gradients and shear stresses generated by the fluid flow causes the surface to erode, which in turn affects the flow structure. This interplay may or may not be incorporated into the hydrodynamic model, but it stands as a criterion in consideration of potential models.

The area of computational fluid dynamics (CFD) has advanced markedly in recent years, driven in part by advances in computational technology of solving the governing Navier-Stokes equations for incompressible flow. There are many and various approaches for solving the Navier-Stokes equations. The first is to solve them directly for specific boundary and initial conditions. This is an ideal approach; the only potential errors in this Direct Numerical Simulation (DNS) method are the ones introduced by the numerical scheme. That is, accuracy is highly dependent on the grid system used and level of spatial-temporal resolution, limiting the approach to simple geometries. But, because it solves the Navier-Stokes equations directly, it gives explicit instantaneous velocities. This is extremely useful for purposes of erosion modelling, in which the entire distributions of velocity values and pressure forces are needed. Due to the physical complexity associated with rough and changing solid surfaces, however, it is unrealistic to employ this approach for erosion modelling.

Secondly, there is Large Eddy Simulation (LES), which filters all instantaneous variables so that they operate at the level of grid resolution or larger, thereby reproducing only the large-scale flow structure. In particular, a turbulent viscosity value is used, which encompasses the range of all viscous forces below the grid resolution scale. This approach gives instantaneous velocity and pressure values that are spatially averaged at the scale of grid cell width. In consideration of the erosion problem, we note that this scale must not be significantly larger than the scale of soil aggregates.

As a third approach, decomposing the instantaneous flow into mean and fluctuating elements and then averaging gives rise to the Reynolds Averaged Navier-Stokes (RANS) equations. This introduces an additional Reynolds stress term, so that some closure model relating this stress to the mean flow is required. The RANS approach is more widely applicable than DNS, in terms of adapting to complex boundary conditions, but depends on the modelling assumptions inherent in the closure scheme. Furthermore, time-averaged velocity profiles are not useful in the context of erosion modelling, since it is primarily the extremes of the pressure distribution that cause soil detachment.

The LES has proven to be a flexible tool for erosion modelling, since it can accommodate complex solid boundaries adequately, it gives full distributions of velocity and pressure, and is compatible with a variety of methods for incorporating sediment dynamics. This modelling method will provide extensive information on velocity and sediment fields, which are needed to underpin the stochastic concept. There are many possibilities for, and difficulties with, implementing the LES method. The key aspect is the representation of the complicated rough boundary, for which there are many approaches, depending on the desired modelling scale. Nonetheless, our methodology for incorporating the results from LES simulations into the stochastic models is straightforward. The output from the LES will be in the form of time series data of velocity and pressure, calculated at each point on the grid. PDFs can then be extracted from spatially averaged time series data by assigning velocity and pressure data to a finite set of bins, and normalizing the frequency at which the data fall into each bin. Other useful statistical constructs, such as structure functions, can also be extracted from the LES time series data.

Soil structure modelling

Soil structure is the spatial/temporal distribution of soil physical characteristics within a soil body. One of these characteristics is the size (linear, by the area; volumetric, by the weight) of soil particles and aggregates. Distribution of soil particles and aggregates by size is described with PDFs, and more recently by fractal dimensions (FDs). These distributions change in time due to fragmentation of soil aggregates or aggregation of soil aggregates and particles. Nevertheless, there are quite a few main types of PDF, estimated empirically and associated with all variety of soils in different conditions. There is the logarithmically Normal distribution, and the Rosin-Rammler relation and power-law distribution, associated with the fractal approach (Perfect *et al.*, 1993). Only the logarithmically Normal distribution has theoretical basis (Kolmogorov, 1941). This work described the process of random failure of soil particles, when the probability of fragmentation of a particle to some number of parts was scale-invariant, and the result was asymptotically logarithmically Normal.

The Kolmogorov-type algorithm of soil particles failure can be simulated numerically, and in numerical experiments the assumption of scale independence of fragmentation can be avoided. These experiments with different relationships between probability of failure and particle size show a great stability of result. The logarithmically Normal distribution of soil particles is valid in a broad range of scenarios of fragmentation. This distribution is asymptotic, but is developed within a first few steps of simulation. Each type of fragmentation process is characterized by specific rates of mean size decrease and particle size variability increase.

NUMERICAL EXPERIMENTS

Numerical experiments were undertaken to show the general advantages of the proposed stochastic approach for soil erosion calculation. To investigate the most important soil erosion factors, the proposed approach was simplified: not all driving and resistance forces were included in the aggregate instability inequality; we only considered lift, gravity and cohesion forces. The stochastic variables in this inequality are assumed to be independent; this allows using the expressions for calculating the probability of the sum and product of independent stochastic variables. Three main types of probability distribution functions for hydrodynamic and soil characteristics were used: Normal, logarithmically Normal and Gamma distribution. The input data consisted of mean bed velocity U_m, mean soil cohesion C_{0m}, mean soil integrity I_s, mean aggregate diameter D_m, and standard deviations for all those variables. Numerical experiments were carried out to analyse the influence of these four

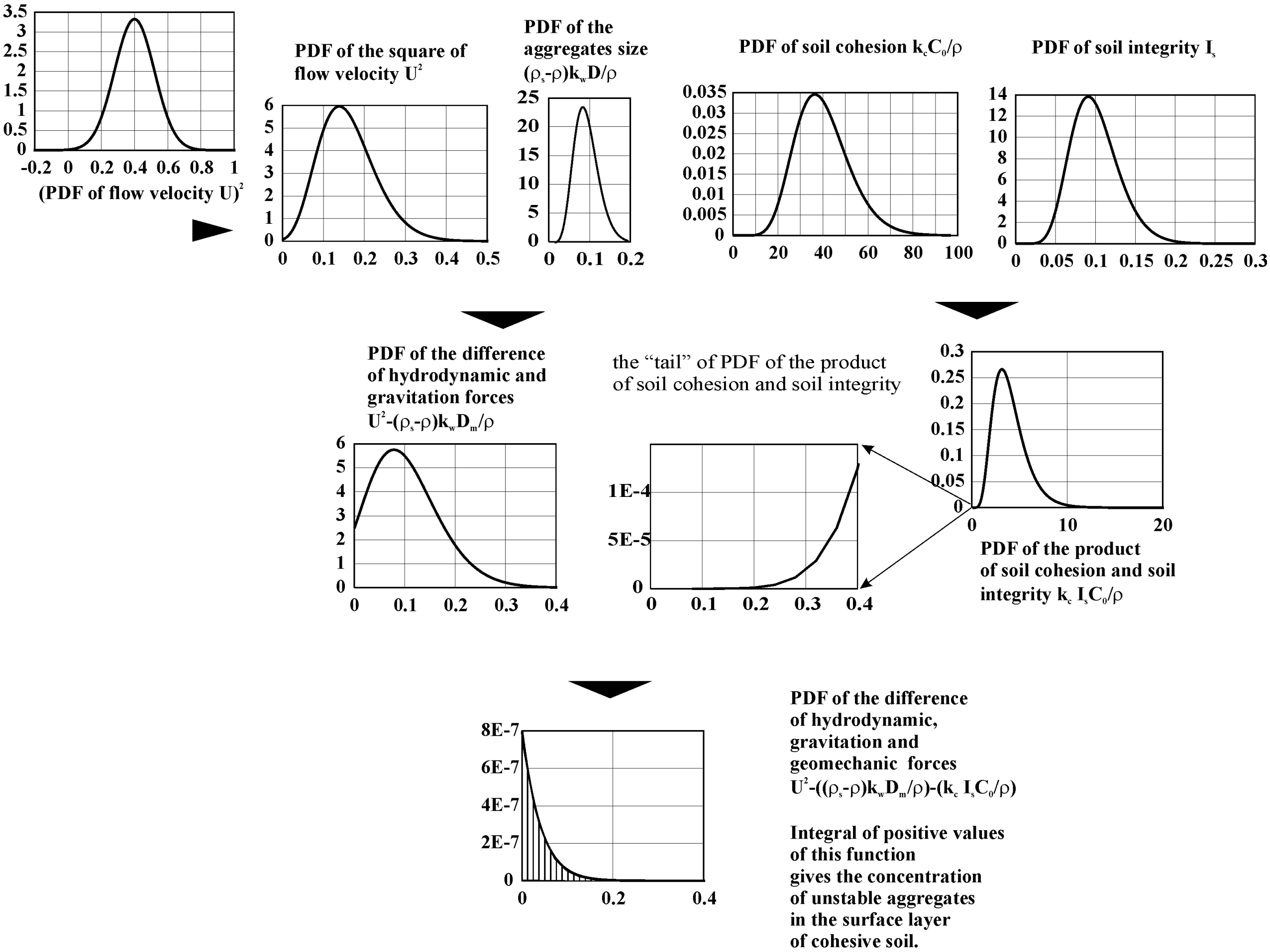

Fig. 1 The order of soil aggregate concentration calculation with the stochastic model.

stochastic factors on the detachment rate. The range of mean flow bed velocity was 0.1–2.2 m s^{-1}, the range of mean cohesion was 1–30 kPa, mean soil integrity ranged from 0.1 to 4, aggregate mean size in the natural soil varied from 1 to 10 mm; and standard deviations for all PDFs varied from 0.1 to 1.0–2.0 times the mean value. The sequence of calculation of sediment concentration of unstable aggregates from PDFs of driving and resistance forces is shown in Fig. 1. The same order is used to obtain the mean vertical velocity and, finally, the detachment rate.

RESULTS AND DISCUSSION

The following main phenomena were observed (Fig. 2):

(a) The increase in erosion rate with flow velocity cannot be described with an often-used simple power function with *a priori* known exponent *n*: $DER \sim U^n$. Calculations show that, when velocities are relatively low, the detachment rate increases more rapidly than in relatively high velocities. A similar effect was described by Nearing *et al.* (1997) on the basis of empirical soil erosion measurements. In this investigation the phenomenon was underpinned theoretically.

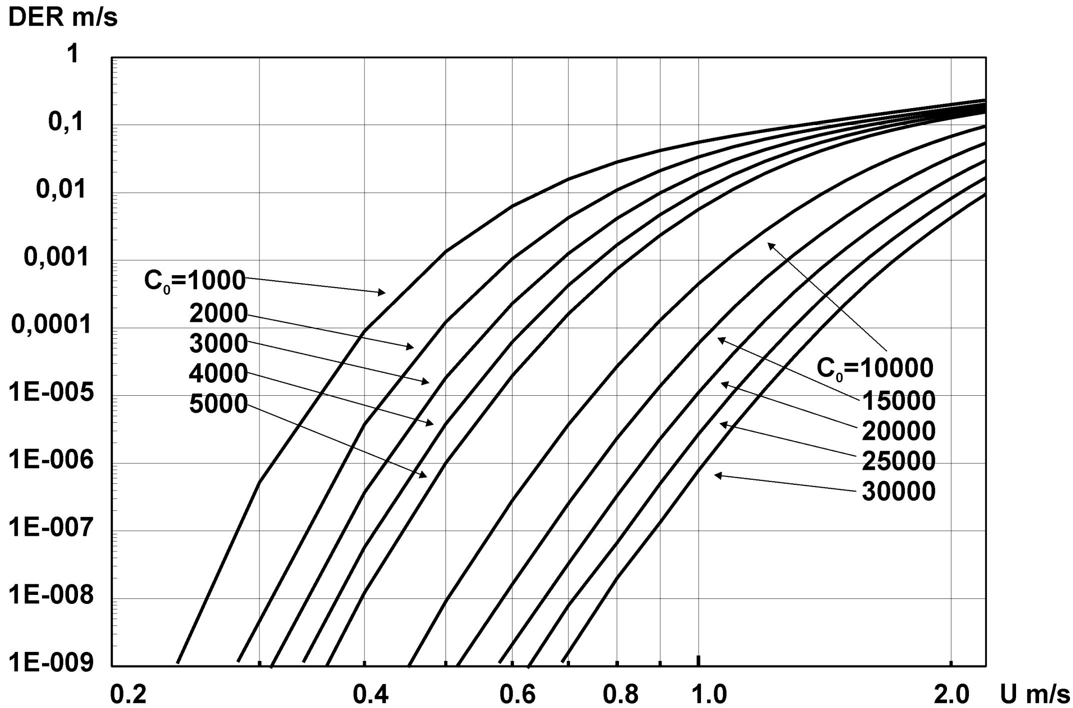

Fig. 2 Relationship between detachment rate (*DER*) and the mean flow velocity (*U*). The third variable is soil cohesion. The calculations were performed with soil integrity 0.1; mean aggregates size 2 mm, variability coefficients for these stochastic variables 0.3.

(b) The analysis of relationships between the hydraulic characteristics of the flow (actual flow velocity), the geo-mechanical properties of the soil (aggregates size, cohesion and integrity), and the soil aggregates detachment rate makes possible an explanation of the difference in relationship types between detachment rate and flow velocity (shear stress, stream power) for different soils. This difference is caused by the relative energy of the flow: the ratio between driving and resistance forces, as well as by the spatial/temporal variability of these forces. In high flow velocities, when driving forces significantly exceed stabilizing forces, the rate of erosion increase with flow velocity is relatively low. The influence of the variability of soil properties (cohesion, aggregate size, and soil integrity) is also less important in determining the soil erosion rate of relatively high flow energy. With low flow velocities and with driving forces only slightly exceeding the stabilizing forces, erosion rates increased rapidly with flow velocity, and all soil properties became sufficient for erosion rate estimation (see Fig. 2 for the influence of soil cohesion, other soil properties give the same effect).

The stochastic erosion models are third-generation models, accepting empirical statistical models (USLE-type) as first-generation models, and shear stress-based models (WEPP-type) as second-generation models. In the new model the relationship between soil detachment rate and the factors of erosion (flow and soil characteristics) is not obtained in advance from some empirical data. They are calculated within the model from the information about PDFs of driving and stabilizing forces with the use of basic equations and are different for the different combinations of erosion factors. Therefore third-generation models promise more precise soil erosion prediction due to more accurate description of soil erosion mechanics, but they require better information about flow and soil.

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