Stochastic Model to predict Water Erosion of Cohesive Soil

Aleksey Sidorchuk[1)]

Abstract.
New method of detachment rate estimation was used in stochastic soil erosion model. It is based on calculation of the probability of excess of driving forces above resistance forces in the flow. Four characteristics are used as stochastic variables: flow velocity, soil cohesion, aggregate size and soil consolidation. The proposed theory allows explicit explanation of a difference in types of relationship between detachment rate and flow velocity. Detachment rate increases with flow velocity more rapidly for more consolidated soil with high cohesion and large aggregates. This theory also shows great difference in type of soil erosion for relatively high and relatively low flow velocities, and explain rather high errors in calculating of soil erosion rate even with the detailed models.
Keywords:Soil erosion, detachment rate, stochastic model

Detachment rate D_r is the product of the concentration C_Δ of active soil aggregates in a near bed layer with thickness Δ on the mean vertical velocity of soil aggregates $U_\uparrow$:

$$D_r = C_\Delta U_\uparrow \qquad (1).$$

Sediment concentration is ratio between volume of soil aggregates and volume of the fluid in the layer Δ for a unit area *S*. Volume of soil aggregates can be written as product of number of active soil aggregates *N* on their mean volume V_m, and unit area can be presented as number of soil aggregates on the unit area *M* on their mean area S_m:

$$C_\Delta = \frac{NV_m}{MS_m\Delta} \qquad (2).$$

The ratio *N/M* is the probability P_d of soil aggregate detachment for a given unit time $dt = \Delta/U_\uparrow$, and the ratio V_m/S_m is a measure of mean soil aggregate height D_m.

$$C_\Delta = P_d \frac{D_m}{\Delta} \qquad (3).$$

A probability of soil aggregate detachment is probability of excess of a driving forces in the flow above a resistance forces. The main driving forces are drag force F_d, lift force F_l, negative turbulent dynamic pressure F_{dp} and pore water pressure F_{pw}. The main resistance forces are submerged aggregate weight F_w, friction force F_f, static pressure F_{sp}, positive turbulent dynamic pressure F_{dp} and cohesion F_c.
A probability of detachment is more than 0, when

$$F_d + F_l + F_{pw} \mp F_{dp} - F_w - F_f - F_{sp} - F_c \ge 0 \qquad (4),$$

or

$$\Psi = U^2 + k_{pw} H_p \frac{S_b}{S_u} \mp k_{dp}\lambda \frac{S_b}{S_u} U_m^2 - k_{wf} D_m \frac{(\rho_s - \rho)}{\rho} - -k_{sp} d \frac{S_b}{S_u} - k_c \frac{C_0}{\rho}\frac{S_b}{S_u} \ge 0 \qquad (5).$$

Here C_R is coefficient of drag resistance; C_y is coefficient of uplift; *U* is actual bottom flow velocity and U_m is its mean value; λ is coefficient of hydraulic resistance; S_d is cross - section area of soil aggregate, perpendicular to flow; ρ_s and ρ are density of soil aggregates (with the pores) and water density; S_u is cross – section area of soil aggregate, parallel to flow; S_b is area of soil aggregate, solid with the soil; H_p is capillary pressure height; f_t is friction coefficient; *d* is water depth; C_0 is soil cohesion; $k_{pw} = \frac{2g}{(kC_R + C_y)} \approx 40$;

$k_{dp} = \frac{3.5}{(kC_R + C_y)} \approx 7$; $k_{wf} = \frac{2(1+f_t)g}{(kC_R + C_y)} \approx 42$; $k_{sp} = \frac{2g}{(kC_R + C_y)} \approx 40$; $k_c = \frac{2}{(kC_R + C_y)} \approx 4$.

Probability of detachment P_d then can be calculated with the formula:

$$P_d = \int_0^\infty p_\Psi d\Psi \qquad (6).$$

Here p_Ψ is function of probability density for a sum of stochastic variables from (5).

1) Doctor of Science (Geography), Geographical Faculty, Moscow State University,
119899 Moscow, Russia, Email sidor@yas.geogr.msu.su

A probability of a function of stochastic variables can be calculated if probabilities of these variables are known. A probability density of product Z of stochastic variables X and Y is derived with the integrals:

$$p_Z(Z) = -\int_{-\infty}^{0} X^{-1} p_X(X) p_Y\left(\frac{Z}{X}\right) dX + \int_{0}^{\infty} X^{-1} p_X(X) p_Y\left(\frac{Z}{X}\right) dX \qquad (7).$$

A probability density of sum Z of explanatory stochastic variables X and Y is derived from resultant of probability functions (Gnedenko, 1954):

$$p_Z(Z) = \int_{-\infty}^{\infty} p_X(X) p_Y(Y) dX = \int_{-\infty}^{\infty} p_X(X) p_Y(Z - X) dX \qquad (8).$$

We shall analyze a simplified case, when four characteristics are stochastic variables: velocity U, cohesion C_0, aggregate size D_m and degree of soil consolidation $I_s=S_b/S_u$, and all others are parameters.

A probability density function for U p_U with mean value U_m and standard deviation σ_U is well known normal distribution. Then frequency of $z=U^2/\sigma^2$ will be defined by first order non - centered χ^2 distribution

$$p_z = \frac{1}{\sqrt{2z}} \exp\left(-\frac{z+m}{2}\right) \sum_{j=0}^{\infty} \frac{(mz/4)^j}{j!\Gamma\left(j+\frac{1}{2}\right)} \qquad (9)$$

Here $m = U_m^2 / \sigma^2$ is parameter of non- – centricity.

Degree of soil consolidation $I_s=S_b/S_u$ is represented by the ratio of area of soil aggregate basement, solid with the soil, to the whole area of an aggregate bottom side. The difference between these two variables is the area of micro - cracks, which cut loose individual aggregate from solid soil. The measure (1- S_b/S_u) is soil fatigue, which is developed in the soil under a dynamic influence of turbulent flow. It is clear that its value varies between 0 and 1, and beta – distribution will be used in further analysis.

$$dp_I = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} (I_s)^{\alpha-1} (1-I_s)^{\beta-1} dI_s \qquad (10)$$

Here Γ is gamma-function. Parameters α and β depend on I_s mean value and standard deviation: $(I_s)_{mean}=\alpha/(\alpha+\beta)$; $\sigma_I = \frac{\sqrt{\alpha\beta/(\alpha+\beta+1)}}{\alpha+\beta}$.

Analysis of Mirtskhulava (1970,1988) data showed that distribution of cohesion within a sample of soil fits to gamma-distribution:

$$dp_C = \frac{\beta^{\alpha}}{\Gamma(\alpha)} C_0^{\alpha-1} \exp(-\beta\, C_0) dC_0 \qquad (11)$$

Parameters α and β are correlated with mean cohesion and standard deviation: $C_m=\alpha/\beta$; $\sigma_C = \sqrt{\alpha/\beta^2}$. Data of Mirtskhulava show that coefficient of variation $C_v=\sigma_C/C_m$ is constant and equals to 0.2 for wide range of soil characteristics, so distribution for actual cohesion is determined only by parameter β or by mean cohesion of these soils.

Distribution density of soil aggregate size within a sample of soil fits to log-normal distribution:

$$p_D = \frac{1}{D\sqrt{2\pi Q}} \exp\left[-\frac{(\ln D - R)^2}{2Q}\right] \quad (12).$$

Parameters Q и R a related to mean aggregate diameter D_m and its standard deviation σ_D : $D_m = \exp\left(\frac{Q}{2} + R\right)$ and $\sigma_D = \sqrt{\exp(Q+2R)[\exp(Q)-1]}$.

Formulas (7)-(8) can be used only for explanatory stochastic variables. To fulfil this condition the order for calculation of probability of detachment may be following: probability density of normalized square of near bed flow velocity $\Psi_1 = U^2/\sigma_U^2$ is calculated with formula (9); probability density of normalized aggregate size $\Psi_2 = k_{wf} D_m \frac{(\rho_s - \rho)}{\sigma_U^2 \rho}$ is calculated with formula (12); probability density of difference of normalized square of near bed flow velocity and aggregate size is calculated with formula $\Psi_3 = \Psi_1 - \Psi_2$ (8); probability density of sum of normalized soil cohesion and static pressure $\Psi_4 = \frac{k_s C_0 + k_{sp} d}{\sigma_U^2 \rho}$ is calculated with formula (11), and non-stochastic components are taken into account; probability density of sum of normalized dynamic and pore pressure $\Psi_5 = \dfrac{k_{pd}\lambda U^2 + k_{pw} H_p}{\sigma_U^2 \rho}$ is calculated with formula (9), and non-stochastic components are taken into account; probability density of difference $\Psi_6 = \Psi_4 - \Psi_5$ is calculated with the formula (8); probability density of product $\Psi_7 = \Psi_6 I_s$ is calculated with the formulas (10) and (7). Final probability density of difference $\Psi = \Psi_3 - \Psi_7$ is calculated with the formula (16). This function is integrated by $d\Psi$ for positive values of Ψ, and probability of soil aggregates detachment and near bed active aggregates concentration is calculated with (6).

Vertical aggregate velocity $U_\uparrow$ is the second component of formula (1) for calculation of detachment rate. At the moment of aggregate detachment its acceleration can be derived from expression

$$k_w D_m \frac{(\rho_s - \rho)}{\sigma_U^2 \rho} \frac{dU_\uparrow}{dt} = k_w D_m \frac{(\rho_s - \rho)}{2\sigma_U^2 \rho} \frac{dU_\uparrow^2}{dz} = \Psi \qquad (13).$$

In the near bed layer of the flow with thickness $\Delta = D_m$ an aggregate accelerates from zero velocity till maximum $U_{\uparrow m}$. With the logarithmic vertical velocity distribution in this layer the integral of (13) gives a simple expression for mean near bed velocity of aggregates:

$$U_\uparrow = \sqrt{\Psi \frac{\Delta}{D_m} \frac{\sigma_U^2 \rho}{\rho_s k_w}} \qquad (14).$$

In turbulent flow with random vertical velocity its mean value at the field of positive sum of forces will be calculated with the formula

$$\overline{U_\uparrow} = \frac{\int_0^\infty p_\Psi \sqrt{\Psi \frac{\Delta}{D_m} \frac{\sigma_U^2 \rho}{(\rho_s - \rho) k_w}}\, d\Psi}{\int_0^\infty p_\Psi \, d\Psi} \qquad (15).$$

Results and Discussion

Theoretical analysis of stochastic mechanics of soil aggregates erosion in water flow shows that in the field of random driving and stabilizing forces the rate of detachment can be calculated as product of (6) and (15):

$$D_r = \int_0^\infty p_\Psi \sqrt{\Psi \frac{\sigma_U^2 \rho}{(\rho_s - \rho) k_w}}\, d\Psi \qquad (16).$$

Analytical form of (16) is rather complicated, and it has to be solved numerically with a certain set of input data. The input data consist of mean bed velocity U_m, mean soil cohesion C_0, mean soil consolidation I_s, mean aggregate diameter D_m and its standard deviation σ_D. Also hydraulic resistance coefficient λ, flow depth d, pore water pressure height H, aggregate density (with porosity) ρ_s have to be known.

Numerical experiments were carried out for analysis of the influence of above mentioned four stochastic factors of detachment rate. The range of flow bed velocity was 0.1-2.0 m/s, the range of cohesion was 1-60 kPa, soil consolidation ranged from 0.1 to 0.9, aggregate mean size – from 1 to10 mm. Other parameters were constant: flow depth was 0.01 mm, pore pressure height – 0.001 m, hydraulic resistance coefficient was 0.01, and aggregate density was 1600 kg/m^3, aggregate size standard deviation was $0.3D_m$.

Detachment rate increases with flow velocity increase (fig.1). This growth of erosion intensity can not be described with often-used simple power function $D_r \sim U_m^n$. Theoretical calculations show that at the field of relatively low velocities detachment rate increases more rapid, than at the field of relatively high velocities. Similar effect was described by M. Nearing et al (1997) on the basement of observations of empirical measurements of soil erosion, and proposed theory explains this phenomenon. Detachment rate growth is controlled by soil cohesion (fig. 1a), by aggregate size (fig. 1b) and, very significantly, by degree of soil consolidation (fig. 1c).

Detachment rate decreases with soil mean cohesion increase. This correlation can not be described with the power functions, the difference with the power law increase with soil cohesion increase and flow velocity decrease. Detachment rate decreases with a degree of soil consolidation increase. This correlation also can not be described with the power law. At the field of relatively high velocities detachment rate decreases slowly, and erosion intensity only slightly depend on soil consolidation. At the field of relatively low velocities the difference from power law became stronger, and soil consolidation became very powerful factor of aggregates detachment rate. The correlation of detachment rate with soil aggregates size is similar to above described correlation with soil consolidation. Detachment rate decreases with soil aggregate size increase, and at the field of relatively high velocities detachment rate decreases slowly, and erosion intensity only slightly depend on soil aggregates size. At the field of relatively low velocities soil aggregates size became pronounced factor of aggregates detachment rate, but not so powerful, as soil consolidation.

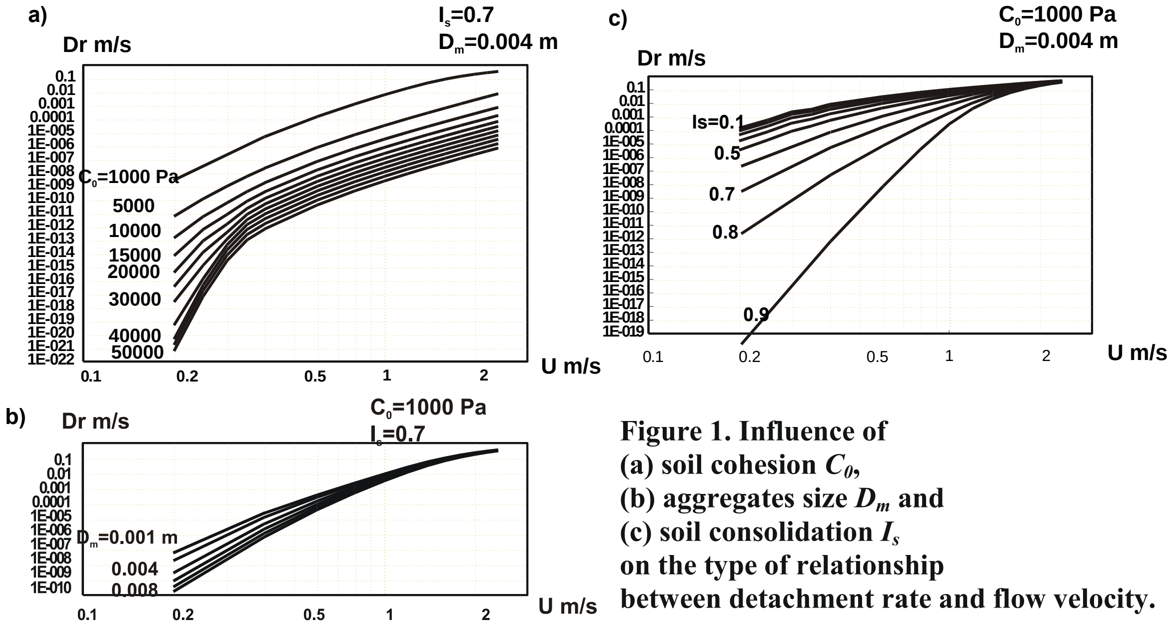

Figure 1. Influence of (a) soil cohesion C_0, (b) aggregates size D_m and (c) soil consolidation I_s on the type of relationship between detachment rate and flow velocity.

Successful verification of these theoretical results was provided by two sets of data: laboratory measurements of Nearing et al. (1991) of the detachment rate for the Russel and Paulding soils, USA, and our field measurements of detachment rate for pebbly loam in Brook Creek gully, Australia (Sidorchuk, 1998). In the both cases optimization calculations were performed to estimate unknown soil consolidation value: for Russel and Paulding soils best fit value of $(I_s)_{mean}$ = 0.8, for pebbly loam in Brook gully this value is 0.1. The same procedure of optimization was used by B. Wilson (1993) for unknown parameters of similar type.

Conclusion

This theory shows great difference in type of soil erosion at the fields of relatively high and relatively low flow velocities. When flow velocities are relatively high and driving forces increase significantly above stabilizing forces, soil properties (cohesion, aggregate size, soil consolidation) are less important in soil erosion. Time and space random variability of these factors, which always exists in natural conditions, does not led to vital changes in erosion rate. This is the main cause of more high predictability of existing soil erosion models for the cases of high-energy events. When flow velocities are relatively low and driving forces only slightly increase above stabilizing forces, soil properties are very important in soil erosion. Even small time and space random variability of these properties led to significant changes in erosion rate. It is objective reason of rather high errors in calculating of soil erosion rate even with the detailed physical based models.

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